Int. J. Solids Structures Vol. 29, No. 14/15, pp. 1883–1888, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00
Pergamon Press plc

AMPLIFICATION OF STRESSES IN THIN LIGAMENTS

XANTHIPPI MARKENSCOFF
Department of Applied Mechanics and Engineering Sciences,
University of California, San Diego, La Jolla, CA 92093, U.S.A.

and

JOHN DUNDURS
Departments of Civil Engineering and Mechanical Engineering,
Northwestern University, Evanston, IL 60208, U.S.A.

Abstract—Stress concentration effects are greatly amplified when cavities approach another boundary and thin ligaments develop. It is shown on the basis of specific solutions that the amplification effect depends not only on the adjacent boundaries forming the ligament, but also on the type of loading, and even on the presence of other far boundaries, when they affect significantly the force and moment transmitted by the ligament.

The goal of theory of elasticity may be to calculate stresses in a piece of material, but one of its aims is also to predict where, how and when something bad, be it through plastic flow or rupture, starts to happen in the material as the intensity of loading is increased. One never-proven conjecture is that the most severe stress conditions occur at the surface. This vague idea is in some sense confirmed by most known solutions and is the *raison d'être* of experimental stress analysis when it relies on measuring surface deformations by use of strain gages, optical techniques or other means. The surface is also the most vulnerable part of the material because the likelihood of flaws is much higher near the surface than further inside. This makes the elastic interaction of cavities, cracks, inclusions and inhomogeneities with a free surface a problem of interest, and there are numerous publications addressing the various phenomena involved. We explore in this note one aspect of such propositions by concentrating on the near interaction with circular cavities and inclusions.

The geometry of a circular boundary near a free surface is natural for bipolar coordinates. The basic formulation of plane elasticity in this coordinate system was given by Jeffery (1920). Jeffery also solved several specific boundary value problems, but apparently made a calculation mistake in dealing with a cavity near a free surface when the far field loading is uniform tension parallel to the free surface (see Fig. 1 for the geometry of the

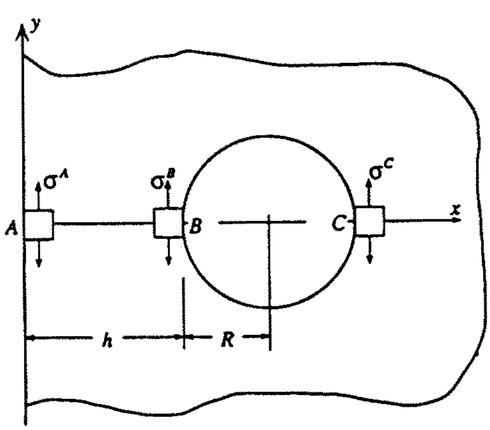

Fig. 1. Geometry and the notation used.

problems discussed in this note). This error was detected and corrected by Mindlin (1948). The Mindlin paper (see his Fig. 10) shows the following behavior of the hoop stresses at the points marked A, B and C in Fig. 1:

(1) First, for very large $\zeta = h/R$,

$$\sigma^A = \sigma_{yy}(0,0) \to T \tag{1}$$

$$\sigma^B = \sigma_{yy}(h,0) \to 3T \tag{2}$$

$$\sigma^C = \sigma_{yy}(h+2R,0) \to 3T, \tag{3}$$

where

$$T = \sigma_{yy}(x, \pm\infty) \tag{4}$$

specifies the intensity of loading. These limits simply confirm the results of the Kirsch solution (Timoshenko and Goodier, 1970, p. 90) for a circular cavity in a field of uniform tension.

(2) In the other extreme, or as $\zeta = h/R \to 0$, his Fig. 10 shows that

$$\sigma^A \to 0 \tag{5}$$

$$\sigma^B \to \infty \tag{6}$$

$$\sigma^C \to 4T, \tag{7}$$

which indicates that, upon increasing the applied load T, fracture would have the tendency to start from the cavity at B and propagate toward the free surface.

The Jeffery–Mindlin solution is in the form of an infinite series, but Mindlin was able to extract the total force transmitted by the neck, or the section AB between the free surface and the circular cavity. The result from his eqn (35) is in the present notation:

$$P = \int_0^h \sigma_{yy}(x,0)\,\mathrm{d}x = Th^{1/2}(2R+h)^{1/2}. \tag{8}$$

A remarkable result in the Mindlin paper is also that it is possible to give a geometric construction that yields the force P in terms of the applied tractions T (see the paragraph below his eqn (35)). Mindlin did not compute the bending moment in the ligament, but in view of his other paper (Mindlin, 1940) dealing with the tunnel problem in bipolar coordinates, it should be possible to do so in closed form.

Equation (8) reveals the following: the force transmitted by the neck vanishes as the cavity approaches the free surface, or when $\zeta = h/R \to 0$. However, the force does not vanish fast enough, and the average stress P/h becomes unbounded. This explains (6), whereas (5) is a special feature of the specific solution. When h is very small in comparison to R, the thin ligament of material acts like a beam with the cross-section AB transmitting the axial force P and a bending moment M. Using the same reasoning as employed in strength of materials, it follows then from (5) and (8) that the asymptotic values for $\zeta \to 0$ are

$$P \sim \sqrt{2}TR\zeta^{1/2} \tag{9}$$

$$M \sim (\sqrt{2}/6)TR^2\zeta^{3/2} \tag{10}$$

$$\sigma^B \sim 2\sqrt{2}T\zeta^{-1/2} + O(1), \tag{11}$$

where

$$M = \int_0^h (x - \tfrac{1}{2}h)\sigma_{yy}(x, 0)\,dx. \tag{12}$$

It is seen that the bending of the ligament is in a direction that makes the free surface concave.

The reciprocal square root dependence of σ^B on $\zeta = h/R$ was first noted by Duan *et al.* (1986). The series in the Jeffery–Mindlin solution do not converge uniformly in the vicinity of $\zeta = 0$, and it is a challenge to extract (11) directly from the series by purely mathematical means. It was shown by Callias and Markenscoff (1989) that this is a problem requiring singular asymptotics. Their analysis confirms (11), also showing that the second term in the asymptotic expansion is of order one. It may be noted that the reciprocal square root dependence shows up also in the interaction of two nearby cavities when the loading is in the direction of the ligament. Again, this was first noted by Duan *et al.* (1986) and later elaborated upon by Zimmerman (1988). However, there is some disagreement in the multiplier of $\zeta^{-1/2}$ between the two papers, and a mathematical analysis using singular asymptotics to establish the multiplier precisely is still missing.

The reciprocal square root dependence in (11) casts a mystique upon the result since it resembles the stress singularity of a Griffith crack. A question then might be whether it is a universal law for the geometry shown in Fig. 1 when the cavity approaches the free surface, similarly to the singularity of a crack which is a universal relation.

This question can immediately be answered in the negative by using another solution worked out by Jeffery (1920), or specifically when the loading is uniform pressure p inside the cavity. This solution is in closed form, and can also be written in terms of elementary biharmonic functions without recourse to bipolar coordinates (Dundurs and Ely, 1965). The key stresses are then

$$\sigma^A = \frac{4p}{\zeta(2+\zeta)} \tag{13}$$

$$\sigma^B = \sigma^C = p \tag{14}$$

for all ζ. The counterparts of (9)–(11) as $\zeta \to 0$ are now

$$P \sim pR \tag{15}$$

$$M \sim -(1/6)pR^2\zeta \tag{16}$$

$$\sigma^A \sim 2p\zeta^{-1} + O(1). \tag{17}$$

The bending of the ligament is opposite to that in the previous case, and the free surface is convex. In contrast to the previous case, (17) indicates that fracture for the pressurized hole would have the tendency to start at the free surface. It may also be noted that there is no essential difference between the pressurized hole and the case when the hole is free of tractions, but the loading is biaxial tension applied both at infinity and on the straight boundary.

Comparison of (17) with (11) shows that the manner in which the stresses in the ligament tend to infinity, as the cavity approaches the free surface, depends entirely on the imposed loading.

Consider next the situation when the circular region of radius R in Fig 1 is filled with material that undergoes thermal expansion. When the material inside R has different elastic constants (inhomogeneity, in the terminology of Eshelby), the problem requires bipolar coordinates. If, however, the elastic constants are the same (inclusion, according to Eshelby), the problem can be solved without recourse to a special coordinate system, because the surrounding material then simply feels a center of dilatation at $x = R+h$ (see the paper by Guell and Dundurs (1967) for an explanation in the case of a spherical inclusion). The solution is in closed form and yields

$$\sigma^{A} = \frac{4p}{(1+\zeta)^2} \tag{18}$$

$$\sigma^{B} = \frac{4p(1+\zeta)(1+\zeta+2\zeta^2)}{(1+2\zeta)^3}. \tag{19}$$

valid for all $\zeta = h/R$. Here p denotes the pressure that the expanding inclusion would exert on the surrounding material if the latter were infinitely extended. The asymptotic limits for $\zeta \to 0$ are now

$$\sigma^{A} = \sigma^{B} \sim 4p + O(\zeta) \tag{20}$$

showing that stresses remain finite no matter how thin the ligament. In the infinitely extended material, there is no difference between an expanding inclusion and a pressurized cavity. Comparison of (20) with (14) and (17) shows that this is not so in the presence of a free surface, and that the behavior is entirely different. It may be of interest to note further that (20) is also valid for an expanding spherical inclusion near a free surface if $4p$ is replaced by $4(1+\nu)p$, where ν stands for Poisson's ratio (Wachtman and Dundurs, 1971).

Apparently the important factor is how the geometry and loading interact, and specifically what is the force and moment transmitted by the thin ligament. The amplification can be seen to be further exacerbated in comparison with the Jeffery–Mindlin problem by considering a central cavity in a strip loaded in tension: if the width of the strip is $2(R+h)$, the force transmitted by each ligament is $T(R+h)$. While in the Jeffery–Mindlin problem the force through the ligament vanishes as the width of the ligament goes to zero, in the strip problem the force transmitted through the ligament remains finite, and the rate of amplification in the two problems is very different. The solution to the strip problem was given by Howland (1930), but the results are virtually intractable for small ζ. However, Koiter (1957) was able to extract the stresses in the ligament using beam theory. His results for $\zeta \to 0$ are

$$\sigma^{A} \sim 0 \tag{21}$$

$$\sigma^{B} \sim 2T\zeta^{-1}, \tag{22}$$

where T is the uniform tensile stress in the strip far away from the hole. Comparison of (22) with (11) shows that the second straight boundary changes the behavior completely.

Finally we touch upon the related issue of two interacting cracks. Some insight into such problems can be gained immediately by considering the simplest possible case of two collinear cracks of equal length in a field of uniform tension. The cracks occupy the intervals $(-b, -a)$ and (a, b) on the x-axis, and the far field loading is $\sigma_{yy}(x, \pm\infty) = T$. This problem was solved by Sadowsky (1956). Sadowsky also gave an explicit expression for the normal stresses transmitted by the neck between the two cracks:

$$\sigma_{yy}(x,0) = T\frac{b^2(E'/K') - x^2}{(a^2-x^2)^{1/2}(b^2-x^2)^{1/2}}, \quad |x| < a, \tag{23}$$

where K' and E' are complete elliptic integrals of the first and second kind to the modulus k', with $k' = (1-k^2)^{1/2}$ and $k = a/b$.

First we estimate asymptotically the force transmitted by the neck

$$P = \int_{-a}^{a} \sigma_{yy}(x,0)\,\mathrm{d}x \tag{24}$$

as $a \to 0$. Noting that

$$K' \sim \frac{\log 4}{k}, \quad E' \sim 1 \tag{25}$$

for $k \to 0$ (Whittaker and Watson, 1958, p. 521), the result is

$$P \sim \frac{\pi T b}{\log (4b/a)}. \tag{26}$$

This shows that the force goes weakly to zero.

From (23):

$$\sigma_{yy}(a^-, 0) = T\frac{b^2(E'/K') - a^2}{(2a)^{1/2}(b^2 - a^2)^{1/2}} \cdot \frac{1}{(a-x)^{1/2}}. \tag{27}$$

For $a \to 0$,

$$\sigma_{yy}(a^-, 0) \sim \frac{Tb}{(2a)^{1/2} \log (4b/a)} \cdot \frac{1}{(a-x)^{1/2}}, \tag{28}$$

which shows that the stress intensity factor increases slightly slower than $a^{-1/2}$.

The stress itself at a generic point in the neck has a different rate of increase. For simplicity take the midpoint between the two cracks ($x = 0$). Then for $a \to 0$,

$$\sigma_{yy}(0, 0) \sim \frac{Tb}{a \log (4b/a)}, \tag{29}$$

showing that stresses inside the neck increase at a slightly slower rate than a^{-1}. By comparing (29) with the $\zeta^{-1/2}$ singularity for the problem of two cavities in a tension field and the ζ^{-1} singularity for a pressurized cavity near a free boundary as discussed previously, it is seen that the cracks have nearly the same behavior as pressurized cavities. This should not be surprising, since there is no essential difference between a crack in a tension field, and a crack that is loaded by pressure applied to its faces.

The following conclusions emerge now from the specific cases considered: the stress concentration effects become strongly amplified as the geometries involved tend to become singular. The amplification depends not only on the geometries of the adjacent boundaries forming a ligament in the limit, but also on the loading conditions and even on other far boundaries, when they affect significantly the force and moment transmitted by the neck of the ligament.

REFERENCES

Callias, C. J. and Markenscoff, X. (1989). Singular asymptotics analysis for the singularity at a hole near a boundary. *Q. Appl. Math.* **47**, 233–245.

Duan, Z. P., Kienzler, R. and Herrmann, G. (1986). An integral equation method and its application to defect mechanics. *J. Mech. Phys. Solids* **34**, 539–561.

Dundurs, J. and Ely, J. F. (1965). Eine Deutung der *Jeffery*schen Lösung für das exzentrische Kreisrohr. *ZAMM* **45**, 262–263.

Guell, D. L. and Dundurs, J. (1967). Further results on center of dilatation and residual stresses in joined elastic half-spaces. In *Developments in Theoretical and Applied Mechanics*, (Edited by W. A. Shaw), Vol. 3, pp. 105–115. Pergamon Press, Oxford.

Howland, R. C. J. (1930). On the stresses in the neighborhood of a circular hole in a strip in tension. *Phil. Trans. R. Soc. Lond.* **A229**, 48–86.

Jeffery, G. B. (1920). Plane stress and plane strain in bipolar co-ordinates. *Phil. Trans. R. Soc. Lond.* **A221**, 265–293.

Koiter, W. T. (1957). Elementary solution of two stress concentration problems in the neighborhood of a hole. *Q. Appl. Math.* **15**, 303–308.

Mindlin, R. D. (1940). Stress distribution around a tunnel. *Trans. ASCE* **105**, 1117–1153.

Mindlin, R. D. (1948). Stress distribution around a hole near the edge of a plate under tension. *Proc. SESA* **5**, 56–67.

Sadowsky, M. (1956). Stress concentration caused by multiple punches and cracks. *J. Appl. Mech.* **23**, 80–84.
Timoshenko, S. P. and Goodier, J. N. (1970). *Theory of Elasticity*, 3rd edn. McGraw-Hill, New York.
Wachtman, J. B. Jr and Dundurs, J. (1971). Large localized surface stresses caused by thermal expansion anisotropy. *J. Am. Ceram. Soc.* **54**, 525–526.
Whittaker, E. T. and Watson, G. H. (1958). *A Course of Modern Analysis*, 4th edn. Cambridge University Press, Cambridge.
Zimmerman, R. W. (1988). Stress singularity around two nearby holes. *Mech. Res. Commun.* **15**, 87–90.